

# Generalizing the variational theory on time scales to include the delta indefinite integral\*

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## Abstract

We prove necessary optimality conditions of Euler–Lagrange type for generalized problems of the calculus of variations on time scales with a Lagrangian depending not only on the independent variable, an unknown function and its delta derivative, but also on a delta indefinite integral that depends on the unknown function. Such kind of variational problems were considered by Euler himself and have been recently investigated in [Methods Appl. Anal. 15 (2008), no. 4, 427–435]. Our results not only provide a generalization to previous results, but also give some other interesting optimality conditions as special cases.

**Keywords:** time scales, calculus of variations, Euler–Lagrange equations, isoperimetric problems, natural boundary conditions.

**Mathematics Subject Classification 2010:** 49K15, 34N05, 39A12.

## 1 Introduction

In what follows,  $\mathbb{T}$  denotes a time scale with operators  $\sigma$ ,  $\rho$ ,  $\mu$ ,  $\nu$ ,  $\Delta$ , and  $\nabla$  [1, 2]. We also assume that there exist at least three points on the time scale:  $a, b, s \in \mathbb{T}$  with  $a < b < s$ , and that the operator  $\sigma$  is delta differentiable. The main purpose of this paper is to generalize the Calculus of Variations on time scales (see [3–8] and references therein) by considering the variational problem

$$\mathcal{L}(y) = \int_a^b L(t, y^\sigma(t), y^\Delta(t), z(t)) \Delta t \longrightarrow \text{extr}, \quad (1)$$

where “extr” denotes “extremize” (i.e., minimize or maximize) and the variable  $z$  in the integrand is itself expressed in terms of an indefinite integral

$$z(t) = \int_a^t g(\tau, y^\sigma(\tau), y^\Delta(\tau)) \Delta \tau.$$

In Subsection 3.1 we obtain the Euler–Lagrange equation for problem (1) in the class of functions  $y \in C_{rd}^1(\mathbb{T}, \mathbb{R})$  satisfying the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta \quad (2)$$

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\*Submitted 25-Jul-2010; revised 27-Nov-2010; accepted 16-Feb-2011; for publication in *Computers & Mathematics with Applications*.

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for some fixed  $\alpha, \beta \in \mathbb{R}$  (cf. Theorem 4). Accordingly to Fraser [9], the idea of generalizing the basic problem of the calculus of variations by considering a variational integral depending also on an indefinite integral (in the classical setting, that is, when  $\mathbb{T} = \mathbb{R}$ ) was first considered by Euler in 1741. Our Euler–Lagrange equation is a generalization of the Euler–Lagrange equations obtained by Euler [9, Eq. (8)], Bohner [3], and Gregory [10]. The transversality conditions for problem (1) are obtained in Subsection 3.2. In Subsection 3.3 we prove a necessary optimality condition for the isoperimetric problem: problem (1)–(2) subject to the delta integral constraint

$$\mathcal{J}(y) = \int_a^b F(t, y^\sigma(t), y^\Delta(t), z(t)) \Delta t = \gamma$$

for some given  $\gamma \in \mathbb{R}$ . In Subsection 3.4 we explain how it is possible to prove backward versions of our results by means of Caputo’s duality [11] (see also [12]). Finally, in Section 4 we provide some applications of our main results.

## 2 Preliminaries

For definitions, notations and results concerning the theory of time scales we refer the readers to the comprehensive books [1, 2]. All the intervals in this paper are time scale intervals. Throughout the text we denote by  $\partial_i f$  the partial derivative of a function  $f$  with respect to its  $i$ th argument.

We assume that

1. the admissible functions  $y$  belong to the class  $C_{rd}^1(\mathbb{T}, \mathbb{R})$ ;
2.  $(t, y, v, z) \rightarrow L(t, y, v, z)$  and  $(t, y, v, z) \rightarrow F(t, y, v, z)$  have continuous partial derivatives with respect to  $y, v, z$  for all  $t \in [a, b]$ ;
3.  $(t, y, v) \rightarrow g(t, y, v)$  has continuous partial derivatives with respect to  $y, v$  for all  $t \in [a, b]$ ;
4.  $t \rightarrow L(t, y^\sigma(t), y^\Delta(t), z(t))$  and  $t \rightarrow F(t, y^\sigma(t), y^\Delta(t), z(t))$  belong to the class  $C_{rd}(\mathbb{T}, \mathbb{R})$  for any admissible function  $y$ ;
5.  $t \rightarrow \partial_3 L(t, y^\sigma(t), y^\Delta(t), z(t))$ ,  $t \rightarrow \partial_3 F(t, y^\sigma(t), y^\Delta(t), z(t))$  and  $t \rightarrow \partial_3 g(t, y^\sigma(t), y^\Delta(t))$  belong to the class  $C_{rd}^1(\mathbb{T}, \mathbb{R})$  for any admissible function  $y$ .

**Definition 1.** An admissible function  $y_* \in C_{rd}^1(\mathbb{T}, \mathbb{R})$  is said to be a local minimizer (resp. local maximizer) to problem (1)–(2) if there exists  $\delta > 0$  such that  $\mathcal{L}(y_*) \leq \mathcal{L}(y)$  (resp.  $\mathcal{L}(y_*) \geq \mathcal{L}(y)$ ) for all admissible  $y$  satisfying the boundary conditions (2) and  $\|y - y_*\| < \delta$ , where

$$\|y\| = \sup_{t \in [a, b]^\kappa} |y^\sigma(t)| + \sup_{t \in [a, b]^\kappa} |y^\Delta(t)|.$$

**Definition 2.** We say that  $\eta \in C_{rd}^1(\mathbb{T}, \mathbb{R})$  is an admissible variation to problem (1)–(2) provided  $\eta(a) = \eta(b) = 0$ .

The following result, known as the *fundamental lemma of the calculus of variations on time scales*, is an important tool in the proofs of our main results. The proof of Lemma 3 follows immediately from [13, Theorem 15] and the duality arguments of Caputo [11].

**Lemma 3.** Let  $f \in C_{rd}([a, b], \mathbb{R})$ . Then

$$\int_a^b f(t) \eta^\sigma(t) \Delta t = 0 \quad \text{for all } \eta \in C_{rd}([a, b], \mathbb{R}) \quad \text{with } \eta(a) = \eta(b) = 0$$

if and only if  $f(t) = 0$  for all  $t \in [a, b]^\kappa$ .

### 3 Main results

In order to simplify expressions, we introduce two operators,  $[\cdot]$  and  $\{\cdot\}$ , defined in the following way:

$$[y](t) := (t, y^\sigma(t), y^\Delta(t), z(t)) \quad \text{and} \quad \{y\}(t) := (t, y^\sigma(t), y^\Delta(t)),$$

where  $y \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ .

#### 3.1 Euler–Lagrange equation

**Theorem 4** (Necessary optimality condition to (1)–(2)). *Suppose that  $y_*$  is a local minimizer or local maximizer to problem (1)–(2). Then  $y_*$  satisfies the Euler–Lagrange equation*

$$\partial_2 L[y](t) - \frac{\Delta}{\Delta t} \partial_3 L[y](t) + \partial_2 g\{y\}(t) \cdot \int_{\sigma(t)}^b \partial_4 L[y](\tau) \Delta \tau - \frac{\Delta}{\Delta t} \left( \partial_3 g\{y\}(t) \cdot \int_{\sigma(t)}^b \partial_4 L[y](\tau) \Delta \tau \right) = 0 \quad (3)$$

for all  $t \in [a, b]^\kappa$ .

*Proof.* Suppose that  $y_*$  is a local minimizer (resp. maximizer) to problem (1)–(2). Let  $\eta$  be an admissible variation and define the function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  by  $\phi(\epsilon) := \mathcal{L}(y_* + \epsilon \eta)$ . It is clear that a necessary condition for  $y_*$  to be an extremizer is given by  $\phi'(0) = 0$ . Note that

$$\begin{aligned} \phi'(0) &= \int_a^b \left( \partial_2 L[y_*](t) \eta^\sigma(t) + \partial_3 L[y_*](t) \eta^\Delta(t) \right. \\ &\quad \left. + \partial_4 L[y_*](t) \cdot \int_a^t (\partial_2 g\{y_*\}(\tau) \eta^\sigma(\tau) + \partial_3 g\{y_*\}(\tau) \eta^\Delta(\tau)) \Delta \tau \right) \Delta t. \end{aligned}$$

Using the integration by parts formula, we obtain

$$\begin{aligned} \int_a^b \partial_3 L[y_*](t) \eta^\Delta(t) \Delta t &= \left[ \partial_3 L[y_*](t) \eta(t) \right]_a^b - \int_a^b \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) \eta^\sigma(t) \Delta t, \\ \int_a^b \left( \partial_4 L[y_*](t) \cdot \int_a^t \partial_2 g\{y_*\}(\tau) \eta^\sigma(\tau) \Delta \tau \right) \Delta t \\ &= \left[ \int_b^t \partial_4 L[y_*](\tau) \Delta \tau \cdot \int_a^t \partial_2 g\{y_*\}(\tau) \eta^\sigma(\tau) \Delta \tau \right]_a^b - \int_a^b \left( \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \cdot \partial_2 g\{y_*\}(t) \eta^\sigma(t) \right) \Delta t \\ &= - \int_a^b \left( \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right) \eta^\sigma(t) \Delta t, \end{aligned}$$

and

$$\begin{aligned} \int_a^b \left( \partial_4 L[y_*](t) \cdot \int_a^t \partial_3 g\{y_*\}(\tau) \eta^\Delta(\tau) \Delta \tau \right) \Delta t \\ &= \left[ \int_b^t \partial_4 L[y_*](\tau) \Delta \tau \cdot \int_a^t \partial_3 g\{y_*\}(\tau) \eta^\Delta(\tau) \Delta \tau \right]_a^b - \int_a^b \left( \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \cdot \partial_3 g\{y_*\}(t) \eta^\Delta(t) \right) \Delta t \\ &= - \int_a^b \left( \partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right) \eta^\Delta(t) \Delta t. \end{aligned}$$

Using again integration by parts in the last integral we obtain

$$\begin{aligned} \int_a^b \left( \partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right) \eta^\Delta(t) \Delta t \\ = \left[ \partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \cdot \eta(t) \right]_a^b - \int_a^b \frac{\Delta}{\Delta t} \left( \partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right) \eta^\sigma(t) \Delta t. \end{aligned}$$

Since  $\eta(a) = \eta(b) = 0$ , then

$$\begin{aligned}\phi'(0) = \int_a^b & \left( \partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right. \\ & \left. + \frac{\Delta}{\Delta t} (\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau) \right) \eta^\sigma(t) \Delta t.\end{aligned}$$

From the optimality condition  $\phi'(0) = 0$  we conclude, by the fundamental lemma of the calculus of variations on time scales (Lemma 3), that

$$\partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau + \frac{\Delta}{\Delta t} \left( \partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right) = 0$$

for all  $t \in [a, b]^\kappa$ , proving the desired result.  $\square$

**Remark 5.** Note that

1. The Euler–Lagrange equation (3) is a generalization of the Euler–Lagrange equation obtained by Euler in 1741 (if  $\mathbb{T} = \mathbb{R}$ , we obtain equation (8) of [9]).
2. Theorem 3.1 of [10] is a corollary of Theorem 4: choose  $g(t, u, v) = u$  and consider the time scale to be the set of real numbers.
3. The Euler–Lagrange equation for the basic problem of the Calculus of Variations on time scales (see, e.g., [3]) is easily obtained from Theorem 4: in this case,  $\partial_4 L = 0$  and therefore we get the equation

$$\partial_2 L(t, y^\sigma(t), y^\Delta(t)) - \frac{\Delta}{\Delta t} \partial_3 L(t, y^\sigma(t), y^\Delta(t)) = 0$$

for all  $t \in [a, b]^\kappa$ .

**Remark 6.** Theorem 4 gives the Euler–Lagrange equation in the delta-differential form. As in the classical case, one can obtain the Euler–Lagrange equation in the integral form. More precisely, the Euler–Lagrange equation in the delta-integral form to problem (1)–(2) is

$$\partial_3 L[y](t) + \partial_3 g\{y\}(t) \cdot \int_{\sigma(t)}^b \partial_4 L[y](\tau) \Delta \tau + \int_t^b \left( \partial_2 L[y](s) + \partial_2 g\{y\}(s) \cdot \int_{\sigma(s)}^b \partial_4 L[y](\tau) \Delta \tau \right) \Delta s = \text{const.}$$

## 3.2 Natural boundary conditions

We now consider the case when the values  $y(a)$  and  $y(b)$  are not necessarily specified.

**Theorem 7** (Natural boundary conditions to (1)). *Suppose that  $y_*$  is a local minimizer (resp. local maximizer) to problem (1). Then  $y_*$  satisfies the Euler–Lagrange equation (3). Moreover,*

1. if  $y(a)$  is free, then the natural boundary condition

$$\partial_3 L[y_*](a) = -\partial_3 g\{y_*\}(a) \cdot \int_{\sigma(a)}^b \partial_4 L[y_*](\tau) \Delta \tau \quad (4)$$

holds;

2. if  $y(b)$  is free, then the natural boundary condition

$$\partial_3 L[y_*](b) = \partial_3 g\{y_*\}(b) \cdot \int_b^{\sigma(b)} \partial_4 L[y_*](\tau) \Delta \tau \quad (5)$$

holds.

*Proof.* Suppose that  $y_*$  is a local minimizer (resp. maximizer) to problem (1). Let  $\eta \in C_{rd}^1(\mathbb{T}, \mathbb{R})$  and define the function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  by  $\phi(\epsilon) := \mathcal{L}(y_* + \epsilon\eta)$ . It is clear that a necessary condition for  $y_*$  to be an extremizer is given by  $\phi'(0) = 0$ . From the arbitrariness of  $\eta$ , and using similar arguments as the ones used in the proof of Theorem 4, we conclude that  $y_*$  satisfies the Euler–Lagrange equation (3).

1. Suppose now that  $y(a)$  is free. If  $y(b) = \beta$  is given, then  $\eta(b) = 0$ ; if  $y(b)$  is free, then we restrict ourselves to those  $\eta$  for which  $\eta(b) = 0$ . Therefore,

$$\begin{aligned} 0 &= \phi'(0) \\ &= \int_a^b \left( \partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right. \\ &\quad \left. + \frac{\Delta}{\Delta t} (\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau) \right) \eta^\sigma(t) \Delta t \\ &\quad - \partial_3 L[y_*](a) \cdot \eta(a) + \partial_3 g\{y_*\}(a) \cdot \int_b^{\sigma(a)} \partial_4 L[y_*](\tau) \Delta \tau \cdot \eta(a). \end{aligned} \quad (6)$$

Using the Euler–Lagrange equation (3) into (6) we obtain

$$\left( -\partial_3 L[y_*](a) + \partial_3 g\{y_*\}(a) \cdot \int_b^{\sigma(a)} \partial_4 L[y_*](\tau) \Delta \tau \right) \cdot \eta(a) = 0.$$

From the arbitrariness of  $\eta$  it follows that

$$\partial_3 L[y_*](a) = \partial_3 g\{y_*\}(a) \cdot \int_b^{\sigma(a)} \partial_4 L[y_*](\tau) \Delta \tau.$$

2. Suppose now that  $y(b)$  is free. If  $y(a) = \alpha$ , then  $\eta(a) = 0$ ; if  $y(a)$  is free, then we restrict ourselves to those  $\eta$  for which  $\eta(a) = 0$ . Thus,

$$\begin{aligned} 0 &= \phi'(0) \\ &= \int_a^b \left( \partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right. \\ &\quad \left. + \frac{\Delta}{\Delta t} (\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau) \right) \eta^\sigma(t) \Delta t \\ &\quad + \partial_3 L[y_*](b) \cdot \eta(b) - \partial_3 g\{y_*\}(b) \cdot \int_b^{\sigma(b)} \partial_4 L[y_*](\tau) \Delta \tau. \end{aligned} \quad (7)$$

Using the Euler–Lagrange equation (3) into (7), and from the arbitrariness of  $\eta$ , it follows that

$$\partial_3 L[y_*](b) = \partial_3 g\{y_*\}(b) \cdot \int_b^{\sigma(b)} \partial_4 L[y_*](\tau) \Delta \tau.$$

□

**Remark 8.** In the classical setting,  $\mathbb{T} = \mathbb{R}$  and  $L$  does not depend on  $z$ . Then, equations (4) and (5) reduce to the well-known natural boundary conditions

$$\partial_3 L(a, y_*(a), y'_*(a)) = 0 \quad \text{and} \quad \partial_3 L(b, y_*(b), y'_*(b)) = 0,$$

respectively.

### 3.3 Isoperimetric problem

We now study the isoperimetric problem on time scales with a delta integral constraint, both for normal and abnormal extremizers. The problem consists of minimizing or maximizing the functional

$$\mathcal{L}(y) = \int_a^b L(t, y^\sigma(t), y^\Delta(t), z(t)) \Delta t, \quad (8)$$

where the variable  $z$  in the integrand is itself expressed in terms of an indefinite delta integral

$$z(t) = \int_a^t g(\tau, y^\sigma(\tau), y^\Delta(\tau)) \Delta \tau,$$

in the class of functions  $y \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ , satisfying the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta \quad (9)$$

and the delta integral constraint

$$\mathcal{J}(y) = \int_a^b F(t, y^\sigma(t), y^\Delta(t), z(t)) \Delta t = \gamma \quad (10)$$

for some given  $\alpha, \beta, \gamma \in \mathbb{R}$ .

**Definition 9.** We say that  $y_* \in C_{rd}^1(\mathbb{T}, \mathbb{R})$  is a local minimizer (resp. local maximizer) to the isoperimetric problem (8)–(10) if there exists  $\delta > 0$  such that  $\mathcal{L}(y_*) \leq \mathcal{L}(y)$  (resp.  $\mathcal{L}(y_*) \geq \mathcal{L}(y)$ ) for all admissible  $y$  satisfying the boundary conditions (9), the isoperimetric constraint (10), and  $\|y - y_*\| < \delta$ .

**Definition 10.** We say that  $y \in C_{rd}^1(\mathbb{T}, \mathbb{R})$  is an extremal to  $\mathcal{J}$  if  $y$  satisfies the Euler–Lagrange equation (3) relatively to  $\mathcal{J}$ . An extremizer (i.e., a local minimizer or a local maximizer) to problem (8)–(10) that is not an extremal to  $\mathcal{J}$  is said to be a normal extremizer; otherwise (i.e., if it is an extremal to  $\mathcal{J}$ ), the extremizer is said to be abnormal.

**Theorem 11** (Necessary optimality condition for normal extremizers of (8)–(10)). Suppose that  $y_* \in C_{rd}^1(\mathbb{T}, \mathbb{R})$  gives a local minimum or a local maximum to the functional  $\mathcal{L}$  subject to the boundary conditions (9) and the integral constraint (10). If  $y_*$  is not an extremal to  $\mathcal{J}$ , then there exists a real  $\lambda$  such that  $y_*$  satisfies the equation

$$\partial_2 H[y](t) - \frac{\Delta}{\Delta t} \partial_3 H[y](t) + \partial_2 g\{y\}(t) \cdot \int_{\sigma(t)}^b \partial_4 H[y](\tau) \Delta \tau - \frac{\Delta}{\Delta t} \left( \partial_3 g\{y\}(t) \cdot \int_{\sigma(t)}^b \partial_4 H[y](\tau) \Delta \tau \right) = 0 \quad (11)$$

for all  $t \in [a, b]^\kappa$ , where  $H = L - \lambda F$ .

*Proof.* Suppose that  $y_* \in C_{rd}^1(\mathbb{T}, \mathbb{R})$  is a normal extremizer to problem (8)–(10). Define the real functions  $\phi, \psi : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$\begin{aligned} \phi(\epsilon_1, \epsilon_2) &= \mathcal{I}(y_* + \epsilon_1 \eta_1 + \epsilon_2 \eta_2), \\ \psi(\epsilon_1, \epsilon_2) &= \mathcal{J}(y_* + \epsilon_1 \eta_1 + \epsilon_2 \eta_2) - \gamma, \end{aligned}$$

where  $\eta_2$  is a fixed variation (that we will choose later) and  $\eta_1$  is an arbitrary variation. Note that

$$\begin{aligned} \frac{\partial \psi}{\partial \epsilon_2}(0, 0) &= \int_a^b \left( \partial_2 F[y_*](t) \eta_2^\sigma(t) + \partial_3 F[y_*](t) \eta_2^\Delta(t) \right. \\ &\quad \left. + \partial_4 F[y_*](t) \cdot \int_a^t (\partial_2 g\{y_*\}(\tau) \eta_2^\sigma(\tau) + \partial_3 g\{y_*\}(\tau) \eta_2^\Delta(\tau)) \Delta \tau \right) \Delta t. \end{aligned}$$

Integration by parts and  $\eta_2(a) = \eta_2(b) = 0$  gives

$$\begin{aligned} \frac{\partial \psi}{\partial \epsilon_2}(0, 0) = \int_a^b \left( \partial_2 F[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 F[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right. \\ \left. + \frac{\Delta}{\Delta t} (\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau) \right) \eta_2^\sigma(t) \Delta t. \end{aligned}$$

Since, by hypothesis,  $y_*$  is not an extremal to  $\mathcal{J}$ , then we can choose  $\eta_2$  such that  $\frac{\partial \psi}{\partial \epsilon_2}(0, 0) \neq 0$ . We keep  $\eta_2$  fixed. Since  $\psi(0, 0) = 0$ , by the implicit function theorem there exists a function  $h$ , defined in a neighborhood  $V$  of zero, such that  $h(0) = 0$  and  $\psi(\epsilon_1, h(\epsilon_1)) = 0$  for any  $\epsilon_1 \in V$ , i.e., there exists a subset of variation curves  $y = y_* + \epsilon_1 \eta_1 + h(\epsilon_1) \eta_2$  satisfying the isoperimetric constraint. Note that  $(0, 0)$  is an extremizer of  $\phi$  subject to the constraint  $\psi = 0$  and

$$\nabla \psi(0, 0) \neq (0, 0).$$

By the Lagrange multiplier rule (cf., e.g., [14]), there exists some constant  $\lambda \in \mathbb{R}$  such that

$$\nabla \phi(0, 0) = \lambda \nabla \psi(0, 0). \quad (12)$$

Since

$$\begin{aligned} \frac{\partial \phi}{\partial \epsilon_1}(0, 0) = \int_a^b \left( \partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right. \\ \left. + \frac{\Delta}{\Delta t} (\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau) \right) \eta_1^\sigma(t) \Delta t \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \psi}{\partial \epsilon_1}(0, 0) = \int_a^b \left( \partial_2 F[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 F[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right. \\ \left. + \frac{\Delta}{\Delta t} (\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau) \right) \eta_1^\sigma(t) \Delta t, \end{aligned}$$

it follows from (12) that

$$\begin{aligned} 0 = \int_a^b \left( \partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right. \\ \left. + \frac{\Delta}{\Delta t} (\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau) \right. \\ \left. - \lambda \left( \partial_2 F[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 F[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right. \right. \\ \left. \left. + \frac{\Delta}{\Delta t} (\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau) \right) \right) \eta_1^\sigma(t) \Delta t. \end{aligned}$$

Using the fundamental lemma of the calculus of variations (Lemma 3), and recalling that  $\eta_1$  is arbitrary, we conclude that

$$\begin{aligned} 0 = \partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \\ + \frac{\Delta}{\Delta t} (\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau) \\ - \lambda \left( \partial_2 F[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 F[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right. \\ \left. + \frac{\Delta}{\Delta t} (\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau) \right) \end{aligned}$$

for all  $t \in [a, b]^\kappa$ , proving that  $H = L - \lambda F$  satisfies the Euler–Lagrange equation (11).  $\square$

**Theorem 12** (Necessary optimality condition for normal and abnormal extremizers of (8)–(10)). *Suppose that  $y_* \in C_{rd}^1(\mathbb{T}, \mathbb{R})$  gives a local minimum or a local maximum to the functional  $\mathcal{L}$  subject to the boundary conditions (9) and the integral constraint (10). Then there exist two constants  $\lambda_0$  and  $\lambda$ , not both zero, such that  $y_*$  satisfies the equation*

$$\partial_2 H[y](t) - \frac{\Delta}{\Delta t} \partial_3 H[y](t) + \partial_2 g\{y\}(t) \cdot \int_{\sigma(t)}^b \partial_4 H[y](\tau) \Delta \tau - \frac{\Delta}{\Delta t} \left( \partial_3 g\{y\}(t) \cdot \int_{\sigma(t)}^b \partial_4 H[y](\tau) \Delta \tau \right) = 0 \quad (13)$$

for all  $t \in [a, b]^\kappa$ , where  $H = \lambda_0 L - \lambda F$ .

*Proof.* Following the proof of Theorem 11, since  $(0, 0)$  is an extremizer of  $\phi$  subject to the constraint  $\psi = 0$ , the abnormal Lagrange multiplier rule (cf., e.g., [14]) guarantees the existence of two reals  $\lambda_0$  and  $\lambda$ , not both zero, such that

$$\lambda_0 \nabla \phi = \lambda \nabla \psi.$$

Therefore,

$$\lambda_0 \frac{\partial \phi}{\partial \epsilon_1}(0, 0) = \lambda \frac{\partial \psi}{\partial \epsilon_1}(0, 0)$$

and hence,

$$\begin{aligned} 0 = & \int_a^b \left( \lambda_0 (\partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right. \\ & + \frac{\Delta}{\Delta t} (\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau)) \\ & - \lambda \left( \partial_2 F[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 F[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right. \\ & \left. \left. + \frac{\Delta}{\Delta t} (\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau) \right) \right) \eta_1^\sigma(t) \Delta t. \end{aligned}$$

From the arbitrariness of  $\eta_1$  and Lemma 3 it is clear that equation (13) holds for all  $t \in [a, b]^\kappa$ , where  $H = \lambda_0 L - \lambda F$ .  $\square$

**Remark 13.** *Note that*

1. *If  $y_*$  is a normal extremizer, then one can consider, by Theorem 11,  $\lambda_0 = 1$  in Theorem 12. The condition  $(\lambda_0, \lambda) \neq (0, 0)$  guarantees that Theorem 12 is a useful necessary condition.*
2. *Theorem 3.4 of [15] is a corollary of our Theorem 11: in that case,  $\partial_4 H = 0$  and we simply obtain*

$$\partial_2 H(t, y^\sigma(t), y^\Delta(t)) - \frac{\Delta}{\Delta t} \partial_3 H(t, y^\sigma(t), y^\Delta(t)) = 0$$

for all  $t \in [a, b]^\kappa$ .

We present two important corollaries that are obtained from Theorem 12 choosing the time scale to be  $\mathbb{T} = h\mathbb{Z} := \{hz : z \in \mathbb{Z}\}$ ,  $h > 0$ , and  $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$ ,  $q > 1$ . In what follows we use the standard notation of *quantum calculus* (see, e.g., [16–18]):

$$\Delta_h y(t) := \frac{y(t+h) - y(t)}{h} \quad \text{and} \quad D_q y(t) := \frac{y(qt) - y(t)}{(q-1)t}.$$

**Corollary 14.** *Let  $h > 0$  and suppose that  $y_*$  is a solution to the discrete-time problem*

$$\mathcal{L}(y) = \sum_{t=a}^{b-h} L(t, y(t+h), \Delta_h y(t), z(t)) \longrightarrow \text{extr}$$



with

$$z(t) = \sum_{\tau=a}^{t-h} g(\tau, y(\tau+h), \Delta_h y(\tau))$$

in the class of functions  $y$  satisfying the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta$$

and the constraint

$$\mathcal{J}(y) = \sum_{t=a}^{b-h} F(t, y(t+h), \Delta_h y(t), z(t)) = \gamma$$

for some given  $\alpha, \beta, \gamma \in \mathbb{R}$ . Then there exist two constants  $\lambda_0$  and  $\lambda$ , not both zero, such that

$$\begin{aligned} 0 = & \partial_2 H(t, y_*(t+h), \Delta_h y_*(t), z_*(t)) - \Delta_h \partial_3 H(t, y_*(t+h), \Delta_h y_*(t), z_*(t)) \\ & + \partial_2 g(t, y_*(t+h), \Delta_h y_*(t)) \cdot \sum_{\tau=t+h}^{b-h} \partial_4 H(\tau, y_*(\tau+h), \Delta_h y_*(\tau), z_*(\tau)) \\ & - \Delta_h \left( \partial_3 g(t, y_*(t+h), \Delta_h y_*(t)) \cdot \sum_{\tau=t+h}^{b-h} \partial_4 H(\tau, y_*(\tau+h), \Delta_h y_*(\tau), z_*(\tau)) \right) \end{aligned}$$

for all  $t \in \{a, a+h, \dots, b-h\}$ , where  $H = \lambda_0 L - \lambda F$ .

*Proof.* Choose  $\mathbb{T} = h\mathbb{Z}$ , where  $a, b \in \mathbb{T}$ . The result follows from Theorem 12.  $\square$

**Corollary 15.** Let  $q > 1$  and suppose that  $y_*$  is a solution to the quantum problem

$$\mathcal{L}(y) = \sum_{t=a}^{bq^{-1}} L(t, y(qt), D_q y(t), z(t)) \longrightarrow \text{extr}$$

with

$$z(t) = \sum_{\tau=a}^{tq^{-1}} g(\tau, y(q\tau), D_q y(\tau))$$

in the class of functions  $y$  satisfying the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta$$

and the constraint

$$\mathcal{J}(y) = \sum_{t=a}^{bq^{-1}} F(t, y(qt), D_q y(t), z(t)) = \gamma$$

for some given  $\alpha, \beta, \gamma \in \mathbb{R}$ . Then there exist two constants  $\lambda_0$  and  $\lambda$ , not both zero, such that

$$\begin{aligned} 0 = & \partial_2 H(t, y_*(qt), D_q y_*(t), z_*(t)) - D_q \partial_3 H(t, y_*(qt), D_q y_*(t), z_*(t)) \\ & + \partial_2 g(t, y_*(qt), D_q y_*(t)) \cdot \sum_{\tau=qt}^{bq^{-1}} \partial_4 H(\tau, y_*(q\tau), D_q y_*(\tau), z_*(\tau)) \\ & - D_q \left( \partial_3 g(t, y_*(qt), D_q y_*(t)) \cdot \sum_{\tau=qt}^{bq^{-1}} \partial_4 H(\tau, y_*(q\tau), D_q y_*(\tau), z_*(\tau)) \right) \end{aligned}$$

for all  $t \in \{a, qa, \dots, bq^{-1}\}$ , where  $H = \lambda_0 L - \lambda F$ .

*Proof.* Choose  $\mathbb{T} = q^{\mathbb{N}_0}$ , where  $a, b \in \mathbb{T}$ . The result follows from Theorem 12.  $\square$

### 3.4 Duality

In the paper [11] (see also [19, 20]) Caputo states that the *delta calculus* and the *nabla calculus* on time scales are the “dual” of each other. A *Duality Principle* is presented, that basically asserts that it is possible to obtain results for the *nabla calculus* directly from results on the *delta calculus* and vice versa. Using the duality arguments of Caputo it is possible to prove easily the *nabla versions* of Theorem 4, Theorem 7, Theorem 11 and Theorem 12.

In what follows we assume that there exist at least three points on the time scale:  $r, a, b \in \mathbb{T}$  with  $r < a < b$ , and that the operator  $\rho$  is nabla differentiable. The following theorem is the *nabla version* of Theorem 12, where the variational problem consists of minimizing or maximizing the functional

$$\mathcal{L}(y) = \int_a^b L(t, y^\rho(t), y^\nabla(t), z(t)) \nabla t, \quad (14)$$

the variable  $z$  in the integrand being itself expressed in terms of a nabla indefinite integral

$$z(t) = \int_a^t g(\tau, y^\rho(\tau), y^\nabla(\tau)) \nabla \tau,$$

in the class of functions  $y \in C_{ld}^1(\mathbb{T}, \mathbb{R})$  satisfying the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta \quad (15)$$

and the nabla integral constraint

$$\mathcal{J}(y) = \int_a^b F(t, y^\rho(t), y^\nabla(t), z(t)) \nabla t = \gamma \quad (16)$$

for some given  $\alpha, \beta, \gamma \in \mathbb{R}$ . We assume that

1. the admissible functions  $y$  belong to the class  $C_{ld}^1(\mathbb{T}, \mathbb{R})$ ;
2.  $(t, y, v, z) \rightarrow L(t, y, v, z)$  and  $(t, y, v, z) \rightarrow F(t, y, v, z)$  have continuous partial derivatives with respect to  $y, v, z$  for all  $t \in [a, b]$ ;
3.  $(t, y, v) \rightarrow g(t, y, v)$  has continuous partial derivatives with respect to  $y, v$  for all  $t \in [a, b]$ ;
4.  $t \rightarrow L(t, y^\rho(t), y^\nabla(t), z(t))$  and  $t \rightarrow F(t, y^\rho(t), y^\nabla(t), z(t))$  belong to the class  $C_{ld}(\mathbb{T}, \mathbb{R})$  for any admissible function  $y$ ;
5.  $t \rightarrow \partial_3 L(t, y^\rho(t), y^\nabla(t), z(t))$ ,  $t \rightarrow \partial_3 F(t, y^\rho(t), y^\nabla(t), z(t))$  and  $t \rightarrow \partial_3 g(t, y^\rho(t), y^\nabla(t))$  belong to the class  $C_{ld}^1(\mathbb{T}, \mathbb{R})$  for any admissible function  $y$ .

The following operators are used:

$$[y](t) := (t, y^\rho(t), y^\nabla(t), z(t)) \quad \text{and} \quad \langle y \rangle(t) := (t, y^\rho(t), y^\nabla(t)), \quad \text{where} \quad y \in C_{ld}^1(\mathbb{T}, \mathbb{R}).$$

**Theorem 16** (Necessary optimality condition for normal and abnormal extremizers of (14)–(16)). *Suppose that  $y_* \in C_{ld}^1(\mathbb{T}, \mathbb{R})$  gives a local minimum or a local maximum to the functional  $\mathcal{L}$  subject to the boundary conditions (15) and the integral constraint (16). Then there exist two constants  $\lambda_0$  and  $\lambda$ , not both zero, such that  $y_*$  satisfies the equation*

$$\partial_2 H[y](t) - \frac{\nabla}{\nabla t} \partial_3 H[y](t) + \partial_2 g\langle y \rangle(t) \cdot \int_{\rho(t)}^b \partial_4 H[y](\tau) \nabla \tau - \frac{\nabla}{\nabla t} \left( \partial_3 g\langle y \rangle(t) \cdot \int_{\rho(t)}^b \partial_4 H[y](\tau) \nabla \tau \right) = 0$$

for all  $t \in [a, b]_\kappa$ , where  $H = \lambda_0 L - \lambda F$ .

**Remark 17.** Theorem 2 of [21] is a corollary of our Theorem 16: in that case  $\partial_4 H = 0$ , and one obtains

$$\partial_2 H(t, y^\rho(t), y^\nabla(t)) - \frac{\nabla}{\nabla t} \partial_3 H(t, y^\rho(t), y^\nabla(t)) = 0$$

for all  $t \in [a, b]_\kappa$ .

From Theorem 4, via duality, one can easily obtain the Euler–Lagrange equation for the nabla problem (14)–(15) (or from Theorem 16 noting that, since there is no nabla integral constraint,  $F = 0$  and  $\gamma = 0$ ).

**Theorem 18** (Necessary optimality condition to (14)–(15)). *Suppose that  $y_*$  is a local minimizer or local maximizer to problem (14)–(15). Then  $y_*$  satisfies the Euler–Lagrange equation*

$$\partial_2 L[y](t) - \frac{\nabla}{\nabla t} \partial_3 L[y](t) + \partial_2 g\langle y \rangle(t) \cdot \int_{\rho(t)}^b \partial_4 L[y](\tau) \nabla \tau - \frac{\nabla}{\nabla t} \left( \partial_3 g\langle y \rangle(t) \cdot \int_{\rho(t)}^b \partial_4 L[y](\tau) \nabla \tau \right) = 0 \quad (17)$$

for all  $t \in [a, b]_\kappa$ .

**Remark 19.** As a corollary of Theorem 18 we obtain the Euler–Lagrange equation for the basic problem of the calculus of variations on nabla calculus [13] (see also [22]). In that case  $\partial L_4 = 0$  and one obtains that

$$\partial_2 L(t, y^\rho(t), y^\nabla(t)) - \frac{\nabla}{\nabla t} \partial_3 L(t, y^\rho(t), y^\nabla(t)) = 0$$

for all  $t \in [a, b]_\kappa$ .

**Remark 20.** Theorem 18 gives the Euler–Lagrange equation in the nabla–differential form. The Euler–Lagrange equation in the nabla–integral form to problem (14)–(15) is

$$\partial_3 L[y](t) + \partial_3 g\langle y \rangle(t) \cdot \int_{\rho(t)}^b \partial_4 L[y](\tau) \Delta \tau + \int_t^b \left( \partial_2 L[y](s) + \partial_2 g\langle y \rangle(s) \cdot \int_{\rho(s)}^b \partial_4 L[y](\tau) \Delta \tau \right) \Delta s = \text{const.}$$

Applying the duality arguments of Caputo to Theorem 7 the following result is obtained.

**Theorem 21** (Natural boundary conditions to (14)). *Suppose that  $y_*$  is a local minimizer (resp. local maximizer) to problem (14). Then  $y_*$  satisfies the Euler–Lagrange equation (17). Moreover,*

1. if  $y(a)$  is free, then the natural boundary condition

$$\partial_3 L[y_*](a) = -\partial_3 g\langle y_* \rangle(a) \cdot \int_{\rho(a)}^b \partial_4 L[y_*](\tau) \Delta \tau$$

holds;

2. if  $y(b)$  is free, then the natural boundary condition

$$\partial_3 L[y_*](b) = -\partial_3 g\langle y_* \rangle(b) \cdot \int_{\rho(b)}^b \partial_4 L[y_*](\tau) \Delta \tau$$

holds.

## 4 Applications

From now on we assume that  $\mathbb{T}$  satisfies the following condition (H):

$$(H) \quad \text{for each } t \in \mathbb{T}, \rho(t) = a_1 t + a_0 \text{ for some } a_1 \in \mathbb{R}^+ \text{ and } a_0 \in \mathbb{R}.$$

**Remark 22.** Note that condition (H) implies that  $\rho$  is nabla differentiable and  $\rho^\nabla(t) = a_1$ ,  $t \in \mathbb{T}_\kappa$ . Also note that condition (H) englobes the differential calculus ( $\mathbb{T} = \mathbb{R}$ ,  $a_1 = 1$ ,  $a_0 = 0$ ), the difference calculus ( $\mathbb{T} = \mathbb{Z}$ ,  $a_1 = 1$ ,  $a_0 = -1$ ), the  $h$ -calculus ( $\mathbb{T} = h\mathbb{Z}$ , for some  $h > 0$ ,  $a_1 = 1$ ,  $a_0 = -h$ ), and the  $q$ -calculus ( $\mathbb{T} = q^{\mathbb{N}_0}$  for some  $q > 1$ ,  $a_1 = \frac{1}{q}$ ,  $a_0 = 0$ ).

The following result illustrates an application of Theorem 16.

**Proposition 23.** Suppose that  $\mathbb{T}$  satisfies condition (H),  $\xi$  is a real parameter, and  $k \in \mathbb{R}$  is a given constant. Suppose that  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a  $C^2$  function that satisfies the conditions:

- (A1)  $\partial_1 f(y^\rho(t), \xi) \neq -ka_1$  for all  $t$  in some non-degenerate interval  $I \subseteq [a, b]$ , for all  $\xi$  and for all admissible function  $y$ ;
- (A2)  $\partial_{1,1}^2 f(y^\rho(t), \xi) \neq 0$  for all  $t$  in some non-degenerate interval  $I \subseteq [a, b]$ , for all  $\xi$  and for all admissible function  $y$ .

Consider

$$L(t, y, v, z) = f(y, \xi) + kz, \quad g(t, y, v) = v \quad \text{and} \quad F(t, y, v, z) = y.$$

If  $y_*$  is a solution to problem (14)–(16), then  $y_*(t) = \alpha$ ,  $t \in [a, b]_\kappa$ .

*Proof.* Suppose that  $y_*$  is an extremizer to problem (14)–(16). By Theorem 16 there exist two constants  $\lambda_0$  and  $\lambda$ , not both zero, such that  $y_*$  satisfies the equation

$$\partial_2 H[y](t) - \frac{\nabla}{\nabla t} \partial_3 H[y](t) + \partial_2 g(y)(t) \cdot \int_{\rho(t)}^b \partial_4 H[y](\tau) \nabla \tau - \frac{\nabla}{\nabla t} \left( \partial_3 g(y)(t) \cdot \int_{\rho(t)}^b \partial_4 H[y](\tau) \nabla \tau \right) = 0 \quad (18)$$

for all  $t \in [a, b]_\kappa$ , where  $H = \lambda_0 L - \lambda F$ . Since

$$\partial_2 H = \lambda_0 \partial_1 f - \lambda, \quad \partial_3 H = 0, \quad \partial_4 H = \lambda_0 k, \quad \partial_2 g = 0 \quad \text{and} \quad \partial_3 g = 1,$$

then equation (18) reduces to

$$\lambda_0 \left( \partial_1 f(y_*^\rho(t), \xi) + ka_1 \right) = \lambda, \quad t \in [a, b]_\kappa. \quad (19)$$

Note that if  $\lambda_0 = 0$ , then  $\lambda = 0$  violates the condition that  $\lambda_0$  and  $\lambda$  do not vanish simultaneously. If  $\lambda = 0$ , then equation (19) reduces to  $\lambda_0 \left( \partial_1 f(y_*^\rho(t), \xi) + ka_1 \right) = 0$ . By assumption (A1) we conclude that  $\lambda_0 = 0$ , which again contradicts the fact that  $\lambda_0$  and  $\lambda$  are not both zero. Consequently, we can assume, without loss of generality, that  $\lambda_0 = 1$ . Hence, equation (19) takes the form

$$\partial_1 f(y_*^\rho(t), \xi) = \lambda - ka_1, \quad t \in [a, b]_\kappa.$$

By assumption (A2) we conclude that

$$y_*^\rho(t) = \text{const}, \quad t \in [a, b]_\kappa.$$

Since  $y(a) = \alpha$ , we obtain that  $y_*(t) = \alpha$  for any  $t \in [a, b]_\kappa$ . □

Observe that the solution to the class of problems considered in Proposition 23 is a constant function that depends only on the boundary conditions (and the isoperimetric constraint) but not explicitly on the integrand function and its parameters.

**Remark 24.** By the isoperimetric constraint (16), a necessary condition for the problem of Proposition 23 to have a solution is that  $\alpha = \frac{\gamma}{b-a}$ .

**Remark 25.** Let  $b$  be a left dense point. Then, by the boundary conditions (15), a necessary condition for the problem of Proposition 23 to have solution is that  $\alpha = \beta$ .

**Remark 26** (cf. [23]). Let  $\mathbb{T} = \mathbb{R}$ . Suppose that  $\alpha = \frac{\gamma}{b-a} = \beta$ .

1. If  $\partial_{1,1}^2 f(y(t), \xi) > 0$  for all  $t \in [a, b]$ , for all  $\xi$  and for all admissible function  $y$ , then problem (14)–(16) has a unique minimizer.
2. If  $\partial_{1,1}^2 f(y(t), \xi) < 0$  for all  $t \in [a, b]$ , for all  $\xi$  and for all admissible function  $y$ , then problem (14)–(16) has a unique maximizer.

We end the paper with an example of application of the *nabla* version of Theorem 11.

**Example 27.** Let  $q : [a, b] \rightarrow \mathbb{R}$  be a continuous function and  $y^{\nabla^2} := (y^{\nabla})^{\nabla}$ . Suppose that  $y_* \in C_{ld}^2$  is an extremizer for

$$\mathcal{L}(y) = \int_a^b \left( (y^{\nabla})^2(t) - q(t)(y^{\rho})^2(t) + 2 \int_a^t y^{\nabla}(\tau) \nabla \tau \right) \nabla t$$

subject to the boundary conditions

$$y(a) = 0 \quad \text{and} \quad y(b) = 0$$

and the delta integral constraint

$$\mathcal{J}(y) = \int_a^b (y^{\rho})^2(t) \nabla t = 1. \quad (20)$$

Note that any extremal to  $\mathcal{J}$  does not satisfy the isoperimetric constraint (20). Hence, this problem has no abnormal extremizers and, by the *nabla* version of Theorem 11, there exists  $\lambda \in \mathbb{R}$  such that  $y_*$  satisfies the equation

$$\partial_2 H[y](t) - \frac{\nabla}{\nabla t} \partial_3 H[y](t) + \partial_2 g(y)(t) \cdot \int_{\rho(t)}^b \partial_4 H[y](\tau) \nabla \tau - \frac{\nabla}{\nabla t} \left( \partial_3 g(y)(t) \cdot \int_{\rho(t)}^b \partial_4 H[y](\tau) \nabla \tau \right) = 0 \quad (21)$$

for all  $t \in [a, b]_{\kappa}$ , where  $H = L - \lambda F$  and

$$L(t, y, v, z) = v^2 - q(t)y^2 + 2z, \quad g(t, y, v) = v, \quad \text{and} \quad F(t, y, v, z) = y^2.$$

Since

$$\partial_2 H = -2qy - 2\lambda y, \quad \partial_3 H = 2v, \quad \partial_4 H = 2, \quad \partial_2 g = 0, \quad \text{and} \quad \partial_3 g = 1,$$

then equation (21) reduces to

$$y^{\nabla^2}(t) + q(t)y^{\rho}(t) + \lambda y^{\rho}(t) = \partial_4 H \cdot \frac{a_1}{2}, \quad t \in [a, b]_{\kappa^2}. \quad (22)$$

Note that in the basic problem of calculus of variations on time scales,  $\partial_4 H = 0$ , and we obtain the *nabla* version of the well known Sturm–Liouville eigenvalue equation:

$$y^{\nabla^2}(t) + q(t)y^{\rho}(t) + \lambda y^{\rho}(t) = 0, \quad t \in [a, b]_{\kappa^2}$$

(see [15, 24]). The study of solutions to equation (22) in the case  $\partial_4 H \neq 0$  is an interesting open problem.

## Acknowledgments

The authors are grateful to the support of the *Portuguese Foundation for Science and Technology* (FCT) through the *Center for Research and Development in Mathematics and Applications* (CIDMA).

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